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Free vibration of deep spherical sandwich shells

S. MIRZA AND A. V. SINGH

Department of Mechanical Engineering, University of Ottawa, Canada

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SUMMARY

This paper deals with axisymmetric vibrations of deep sandwich spherical shells. The shell consists of a thick core and two face sheets of the same isotropic material with equal thickness. The appropriate differential equations have been obtained in terms of two new arbitrary functions which replace the meridional displacement components. The solution is in terms of Legendre functions from which numerical results are computed by using an iterative technique. Effects of geometric and elastic properties of the core and the face sheets are presented in the form of graphs.

1. Introduction

In recent years considerable interest has been shown by many investigators in the dynamic behavior of sandwich plates and shells. With the exception of a few, most of the papers available in literature deal with the vibration of homogeneous spherical shells [1, 4, 5] and shallow spherical sandwich shells [2, 3]. The free vibrations of shallow and deep homogeneous shells have been studied quite intensively by many authors and is available in literature. Very little work has been published on the vibrations of spherical sandwich shells. Koplik and Yu [2] have solved the problem of axisymmetric vibrations of spherical sandwich caps using the associated variational equations of motion and studied the influence of curvature as well as that of the thickness shear deformation on the natural frequencies. Mirza and Doige [3] presented the transverse vibrational characteristics of a three-layered shallow spherical sandwich shell. In their analysis, the longitudinal inertia terms have been neglected and first three modes of the frequency response are shown.

In this paper, axisymmetric vibrations of nonshallow spherical sandwich shells, closed at one pole and open at the other, have been studied. The shell consists of a thick core and of two face sheets of the same isotropic material with equal thickness. The core is assumed to be incompressible in the radial direction and the effect of thickness shear deformation has been considered. The face parallel stresses in the core are assumed to be negligible and the faces are taken as membranes. The complete system of differential equations, based on linear strain-displacement relationships, has been derived from the general equations of motion in a continuum. The appropriate form of the differential equations is achieved by introducing two new arbitrary functions in terms of the meridional displacement components.

The general solution of the differential equations has been expressed in terms of Legendre functions of real as well as complex orders. The highly transcendental equations are solved on the digital computer using an iterative technique. The effects of the geometric and elastic properties of the core and the face sheets are presented in the form of graphs.

2. Formulation of the problem

The equations of motion in a continuum for the axisymmetric case in spherical coordinates r, ϕ, θ can be written as [6]

$$\begin{aligned}(r^2\sigma_r)_{,r}+r\tau_{r\phi,\phi}+r(-\sigma_\phi-\sigma_\theta+\cot\phi\,\tau_{r\phi}) &= \rho r^2 w_{,tt}\\ (r^2\tau_{r\phi})_{,r}+r\sigma_{\phi,\phi}+r\{\tau_{r\phi}+(\sigma_\phi-\sigma_\theta)\cot\phi\} &= \rho r^2 v_{,tt}\end{aligned} \tag{2.1}$$

where σ_r, σ_ϕ and σ_θ are the normal stresses and $\tau_{r\phi}$ is the shear stress. Also, v and w represent

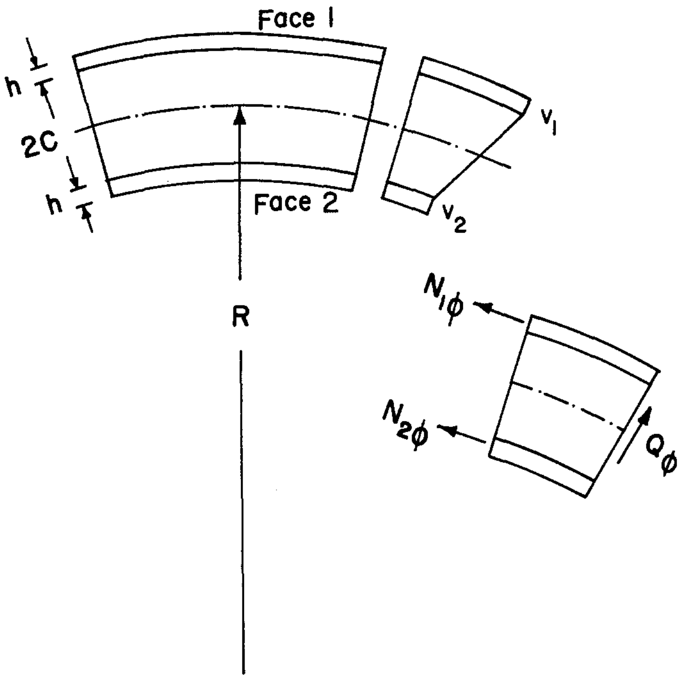

Figure 1. Sandwich shell elements showing forces and variation of displacement.

the components of displacement in ϕ and r directions respectively. The normal strain components ε_r, ε_ϕ, and ε_θ and the shear strain $\gamma_{r\phi}$ are given as

$$\begin{aligned} \varepsilon_r &= w_{,r} \\ \varepsilon_\phi &= (v_{,\phi}+w)/r \\ \varepsilon_\theta &= (v \cot \phi + w)/r \\ \gamma_{r\phi} &= (rv_{,r}-v+w_{,\phi})/r \end{aligned} \tag{2.2}$$

In eqs. (2.1) and (2.2), the notation used for differentiation is

$$\partial w/\partial r = w_{,r}\,, \text{ etc.}$$

Weak core

The assumptions are

(i) $\sigma_\phi = \sigma_\theta = \tau_{\phi\theta} = 0$ (2.3a)

(ii) The core is incompressible in the radial direction,

$$w_1 = w_2 = w_c = w \tag{2.3b}$$

(iii) The variation of the meridional displacement along the thickness of the core is taken in the form

$$v = v^* + \frac{z}{c}\,\bar{v} \tag{2.3c}$$

where

$$\begin{aligned} v^* &= (v_1+v_2)/2 \\ \bar{v} &= (v_1-v_2)/2 \end{aligned} \tag{2.3d}$$

The quantities v_1 and v_2 are the meridional displacements for face sheet 1 and 2 respectively and $2c$ is thickness of the core. The variable z represents the distance measured from the middle surface of the core in the radial direction. Integrating eq. (2.1) between $z = \pm c$ and using eqs. (2.2) and (2.3), the following equations are obtained.

$$(R+z)^2 \sigma_r|_{z=c} - (R+z)^2 \sigma_r|_{z=-c} + 2cG_c \left\{ w_{,\phi\phi} + \cot\phi\, w_{,\phi} - v^*_{,\phi} - \cot\phi\, v^* + \frac{R}{c}(\bar{v}_{,\phi} + \cot\phi\, \bar{v}) \right\} = 2c\rho_c R^2 (1 + \tfrac{1}{3}c^2/R^2) w_{,tt} \tag{2.4}$$

$$(R+z)^2 \tau_{r\phi}|_{z=c} - (R+z)^2 \tau_{r\phi}|_{z=-c} + 2cG_c \left\{ w_{,\phi} - v^* + \frac{R}{c}\bar{v} \right\} = 2c\rho_c R^2 \left\{ (1 + \tfrac{1}{3}c^2/R^2) v^*_{,tt} + \frac{2c}{3R}\bar{v}_{,tt} \right\}.$$

The density of the core and its shear modulus are designated by ρ_c and G_c respectively.

Face layers

The stress resultants in the ith face sheet of thickness h are defined in the following manner

$$N_{\phi i} = \int_{-h/2}^{h/2} (1 + z_i/R_i) \sigma_{\phi i}\, dz_i$$
$$N_{\theta i} = \int_{-h/2}^{h/2} (1 + z_i/R_i) \sigma_{\theta i}\, dz_i \tag{2.5}$$
$$Q_{\phi i} = \int_{-h/2}^{h/2} (1 + z_i/R_i) \tau_{r\phi i}\, dz_i .$$

Use of eqs. (2.2) and the Hooke's Law yields stress resultants in terms of displacements for the ith face sheet

$$N_{\phi i} = \frac{K}{R_i}\{v_{i,\phi} + w + v(v_i \cot\phi + w)\} \quad N_{\theta i} = \frac{K}{R_i}\{v_i \cot\phi + w + v(v_{i,\phi} + w)\} \tag{2.6}$$

where $K = Eh/(1-v^2)$ and E and v are modulus of elasticity and Poisson's ratio in order. For free vibrations of the shell, its outer surfaces are stress free. Thus

$$\sigma_{r_1}|_{z_1=h/2} = \sigma_{r_2}|_{z_2=-h/2} = 0$$
$$\tau_{r\phi_1}|_{z_1=h/2} = \tau_{r\phi_2}|_{z_2=-h/2} = 0 \tag{2.7}$$

The radii of the middle surface of the face sheets are written as

$$R_1 = R + \delta$$
$$R_2 = R - \delta \tag{2.8}$$

where

$$\delta = c + h/2 \tag{2.9}$$

The equations for the face sheets, as given below, are obtained by the introduction of eqs. (2.7) and (2.8) in the integrated form of eq. (2.1)

$$-(R_1+z_1)^2 \sigma_{r_1}|_{z_1=-h/2} - K(1+v)(v_{1,\phi} + \cot\phi\, v_1 + 2w) = \rho h R^2 (1 + \delta^2/R^2 + 2\delta/R) w_{,tt}$$
$$-(R_1+z_1)^2 \tau_{r\phi_1}|_{z_1=-h/2} + K\{v_{1,\phi\phi} + \cot\phi\, v_{1,\phi} - \operatorname{cosec}^2\phi\, v_1 + (1-v)v_1 + (1+v)w_{1\phi}\} = \rho h R^2 (1 + \delta^2/R^2 + 2\delta/R) v_{1,tt}$$
$$(R_2+z_2)^2 \sigma_{r_2}|_{z_2=h/2} - K(1+v)(v_{2,\phi} + \cot\phi\, v_2 + 2w) = \rho h R^2 (1 + \delta^2/R^2 - 2\delta/R) w_{,tt}$$
$$(R_2+z_2)^2 \tau_{r\phi_2}|_{z_2=h/2} + K\{v_{2,\phi\phi} + \cot\phi\, v_{2,\phi} - \operatorname{cosec}^2\phi\, v_2 + (1-v)v_2 + (1+v)w_{,\phi}\} = \rho h R^2 (1 + \delta^2/R^2 - 2\delta/R) v_{2,tt} . \tag{2.10}$$

Equations for composite shell

To establish governing differential equations, eqs. (2.4) and (2.10) are combined by suitable addition and subtraction. The stresses at the interfaces are eliminated by using the following stress continuity conditions.

$$\sigma_r\big|_{z=c} = \sigma_{r_1}\big|_{z_1=-h/2}\ ; \quad \sigma_r\big|_{z=-c} = \sigma_{r_2}\big|_{z_2=h/2}$$
$$\tau_{r\phi}\big|_{z=c} = \tau_{r\phi_1}\big|_{z_1=-h/2}\ ; \quad \tau_{r\phi}\big|_{z=-c} = \tau_{r\phi_2}\big|_{z_2=h/2} \tag{2.11}$$

The summation of eqs. (2.4a), (2.10a and c) yields

$$r_G r_h(1-\nu^2)(w_{,\phi\phi}+\cot\phi\, w_{,\phi}) - \{r_G r_h(1-\nu^2)+1+\nu\}(v^*_{,\phi}+$$
$$+\cot\phi\, v^*) + r_G r_h(1-\nu^2)\frac{R}{c}(\bar{v}_{,\phi}+\cot\phi\,\bar{v}) - 2(1+\nu)w = (1-\nu^2)\rho R^2 s_1 w_{,tt}/E \tag{2.12a}$$

Similarly, eq. (2.12b) is obtained from eqs. (2.4b), (2.10b and d).

$$\{r_G r_h(1-\nu^2)+1+\nu\}w_{,\phi} + v^*_{,\phi\phi} + \cot\phi\, v^*_{,\phi} - \operatorname{cosec}^2\phi\, v^* + \{1-\nu-r_G r_h(1-\nu^2)\}v^*$$
$$+r_G r_h(1-\nu^2)R\bar{v}/c = \rho R^2(1-\nu^2)\{s_1 v^*_{,tt} + s_2 \bar{v}_{,tt}\}/E \tag{2.12b}$$

Subtraction of eq. (2.10d) from (2.10b) and substitution of the values of $\tau_{r\phi}$ at $z = \pm c$ results in

$$-r_G(1-\nu^2)\frac{R}{h}w_{,\phi} + r_G(1-\nu^2)\frac{R}{h}v^* + \bar{v}_{,\phi\phi} + \cot\phi\,\bar{v}_{,\phi} - \operatorname{cosec}^2\phi\,\bar{v} +$$
$$+\left\{(1-\nu) - r_G(1-\nu^2)\frac{R^2}{ch}\right\}\bar{v} = \rho R^2(1-\nu^2)\{s_3 v^*_{,tt} + s_4\bar{v}_{,tt}\}/E \tag{2.12c}$$

In the above equations

$$r_G = G_c/E\,,\quad r_\rho = \rho_c/\rho\,,\quad r_h = c/h \qquad s_2 = 2\delta/R + \tfrac{2}{3}r_\rho r_h c/R$$
$$s_1 = 1+\delta^2/R^2 + r_\rho r_h(1+\tfrac{1}{3}c^2/R^2) \qquad s_3 = 2\delta/R \quad\text{and}\quad s_4 = 1+\delta^2/R^2 \tag{2.13}$$

3. Solution of the differential equations

The solution of eqs. (2.12) for the free axisymmetric vibrations of deep shell is sought in the form

$$w = e^{i\omega t}W\,,\quad v^* = e^{i\omega t}V^*\,,\quad \bar{v} = e^{i\omega t}\bar{V} \tag{3.1}$$

The quantities θ^* and $\bar{\theta}$ are introduced such that

$$\theta^* = V^*_{,\phi} + \cot\phi\, V^* \qquad \bar{\theta} = \bar{V}_{,\phi} + \cot\phi\,\bar{V} \tag{3.2}$$

Equations (2.12) are simplified by the use of eqs. (3.1) and (3.2)

$$A_1(W_{,\phi\phi}+\cot\phi\, W_{,\phi}) + A_2 W + A_3\theta^* + A_4\bar{\theta} = 0$$
$$B_1 W_{,\phi} + \theta^*_{,\phi} + B_2 V^* + B_3\bar{V} = 0 \tag{3.3}$$
$$C_1 W_{,\phi} + C_2 V^* + \bar{\theta}_{,\phi} + C_3\bar{V} = 0$$

where,

$$A_1 = r_G r_h(1-\nu^2)\,, \qquad A_2 = \Omega^2 s_1(1-\nu^2) - 2(1+\nu)$$
$$A_3 = -\{r_G r_h(1-\nu^2)+1+\nu\}\,, \qquad A_4 = r_G r_h(1-\nu^2)R/c$$
$$B_1 = r_G r_h(1-\nu^2)+1+\nu$$
$$B_2 = 1-\nu-r_G r_h(1-\nu^2)+\Omega^2 s_1(1-\nu^2) \tag{3.4}$$
$$B_3 = r_G r_h(1-\nu^2)R/c + \Omega^2 s_2(1-\nu^2)$$
$$C_1 = -r_G(1-\nu^2)R/h\,, \qquad C_2 = r_G(1-\nu^2)R/h + \Omega^2(1-\nu^2)s_3$$
$$C_3 = 1-\nu+\Omega^2 s_4(1-\nu^2) - r_G(1-\nu^2)R^2/ch$$

and Ω is a nondimensional frequency parameter given by

$$\Omega^2 = \rho\omega^2 R^2/E \tag{3.5}$$

Equations (3.3) can be further simplified to the form shown below,

$$\begin{aligned} &(A_1\nabla^2 + A_2)W + A_3\theta^* + A_4\bar{\theta} = 0 \\ &B_1\nabla^2 W + (\nabla^2 + B_2)\theta^* + B_3\bar{\theta} = 0 \\ &C_1\nabla^2 W + C_2\theta^* + (\nabla^2 + C_3)\bar{\theta} = 0 \end{aligned} \tag{3.6}$$

Here ∇^2 is the Laplace operator given by

$$\nabla^2 = \partial^2/\partial\phi^2 + \cot\phi\,\partial/\partial\phi$$

The general solution of eqs. (3.6) can be expressed in terms of Legendre functions of degree n_α ($\alpha = 1, 2, 3$). For a sandwich spherical shell closed at one pole and open at the other, W, θ^* and $\bar{\theta}$ are assumed as,

$$\begin{aligned} \theta^* &= \sum_{\alpha=1}^{3} A_{n_\alpha} P_{n_\alpha}(\cos\phi) \\ \bar{\theta} &= \sum_{\alpha=1}^{3} B_{n_\alpha} P_{n_\alpha}(\cos\phi) \\ W &= \sum_{\alpha=1}^{3} C_{n_\alpha} P_{n_\alpha}(\cos\phi) \end{aligned} \tag{3.7}$$

For brevity, a parameter λ_α is introduced such that

$$n_\alpha(n_\alpha+1) = \lambda_\alpha \tag{ii}$$

which yields

$$n_\alpha = (\lambda_\alpha + \tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2} \tag{iii}$$

Replacing eqs. (3.7) in eqs. (3.6), a system of equations is obtained in terms of the frequency and the unknown coefficients. For a nontrivial solution of A_{n_α}, B_{n_α}, and C_{n_α}, the determinant of the coefficient matrix must vanish, leading to the following characteristic equation.

$$A_1\lambda_\alpha^3 + a_2\lambda_\alpha^2 + a_3\lambda_\alpha + a_4 = 0 \tag{3.8}$$

where,

$$\begin{aligned} a_2 &= -A_2 + A_3B_1 + A_4C_1 - A_1B_2 - A_1C_3 \\ a_3 &= A_1B_2C_3 - A_1B_3C_2 + A_2B_2 + A_2C_3 + A_3B_3C_1 - A_3B_1C_3 + A_4B_1C_2 - A_4B_2C_1 \\ a_4 &= A_2B_3C_2 - A_2B_2C_3 \end{aligned}$$

The displacements V^* and $\bar{V}$ are obtained by substituting eqs. (3.7) in eqs. (3.3b and c) and simplifying them with the help of eqs. (3.6)

$$\begin{aligned} V^* &= \sum_{\alpha=1}^{3} \beta_\alpha \{P_{n_\alpha}(\cos\phi)\}_{,\phi} C_{n_\alpha} \\ \bar{V} &= \sum_{\alpha=1}^{3} \eta_\alpha \{P_{n_\alpha}(\cos\phi)\}_{,\phi} C_{n_\alpha} \end{aligned} \tag{3.9}$$

The parameters β_α and η_α are given as

$$\begin{aligned} \beta_\alpha &= (B_1\lambda_\alpha + B_3C_1 - B_1C_3)/\{\lambda_\alpha^2 - (B_2+C_3)\lambda_\alpha + B_2C_3 - B_3C_2\} \\ \eta_\alpha &= (C_1\lambda_\alpha + B_1C_2 - B_2C_1)/\{\lambda_\alpha^2 - (B_2+C_3)\lambda_\alpha + B_2C_3 - B_3C_2\} \end{aligned}$$

4. Frequency equations

The boundary conditions for the clamped edge are written as

$$W = V^* = \bar{V} = 0 \quad \text{at} \quad \phi = \phi_0 \tag{4.1}$$

These three conditions lead to three equations in terms of the constants C_{n_α} ($\alpha = 1, 2, 3$)

The frequency equation in general is obtained by making the determinant of the coefficient matrix of C_{n_α} equal to zero. Symbolically it gives,

$$|D_{i\alpha}| = 0 \qquad (i, \alpha = 1, 2, 3) \tag{4.2}$$

The exact values of the elements $D_{i\alpha}$, which are obtained from the boundary conditions prescribed at the edge $\phi = \phi_0$, are given as

$$\begin{aligned} D_{1\alpha} &= P_{n_\alpha}(\cos \phi_0) \\ D_{2\alpha} &= \beta_\alpha \{P_{n_\alpha}(\cos \phi_0)\}_{,\phi} \\ D_{3\alpha} &= \eta_\alpha \{P_{n_\alpha}(\cos \phi_0)\}_{,\phi} \,; \qquad (\alpha = 1, 2, 3) \end{aligned} \tag{4.3}$$

5. Numerical computation and discussion of the results

The numerical computations have been carried out for the nondimensional frequency para-

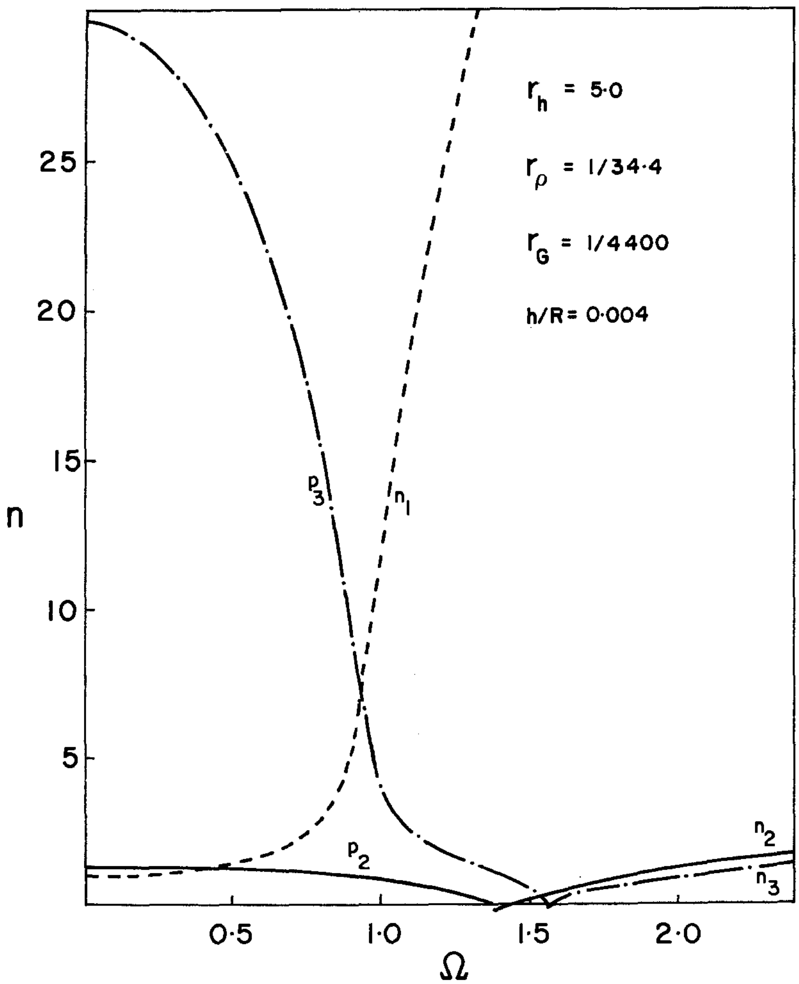

Figure 2. Orders of Legendre functions vs Ω.

meter Ω of the sandwich shell. The variation of Ω has been studied with the geometric and elastic properties of the core and the face sheets. Since the analytical closed form solutions of highly transcendental eqs. (4.2) occurring in the frequency computation is not possible, high speed digital computer has been used to solve the equations by means of an iteration process.

In order to generate the numerical results, the quantities r_G, r_ρ and r_h which are defined in eq. (2.13) are selected as, $r_G = 1/4400$, $r_\rho = 1/34.4$, $r_h = 5$. The ratio h/R is varied over the range 0.001 to 0.005 and the Poisson's ratio $\nu = 0.3$. The quantities $A_1, A_2, \ldots, C_2, C_3$ given by eqs. (3.4) are the functions of these parameters. At this stage there remain only two unknowns λ_α and Ω which are obtained from the eqs. (3.8) and (4.2). By specifying the values of Ω, the roots λ_α ($\alpha = 1, 2, 3$) of the characteristic cubic eq. (3.8) are calculated. Since the coefficients of eq. (3.8) are real, it always generates one real root and the other two are either real or complex conjugates. The orders of Legendre functions occurring in the frequency determinant (4.2) are evaluated with the help of eq. (iii) and the values of λ_α. In the case, when the roots of the characteristic cubic equation are real, the order of the Legendre function n_α ($\alpha = 1, 2, 3$) are either real or of the form

$$n_\alpha = -\tfrac{1}{2} + ip_\alpha$$

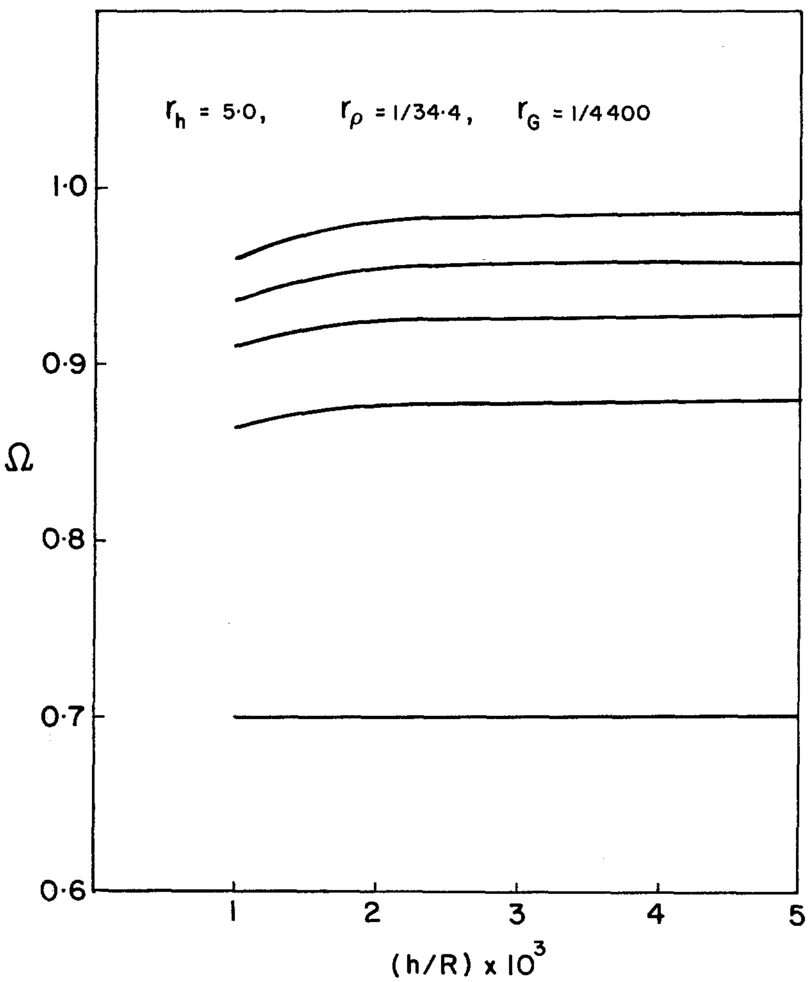

Figure 3. Frequency parameter Ω vs h/R for clamped edge at $\phi = 90°$.

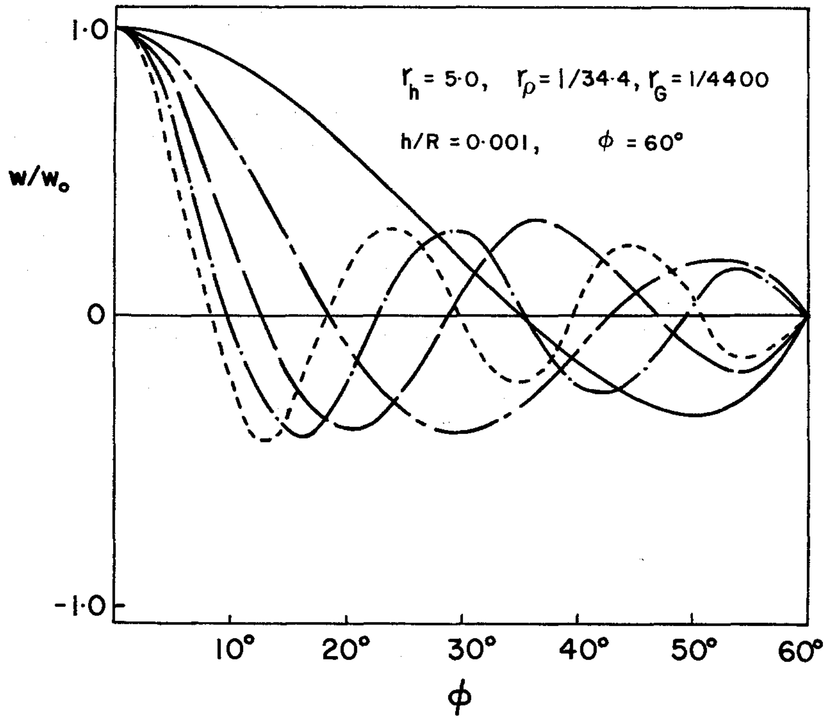

Figure 4. Mode shapes for clamped edge spherical sandwich shell.

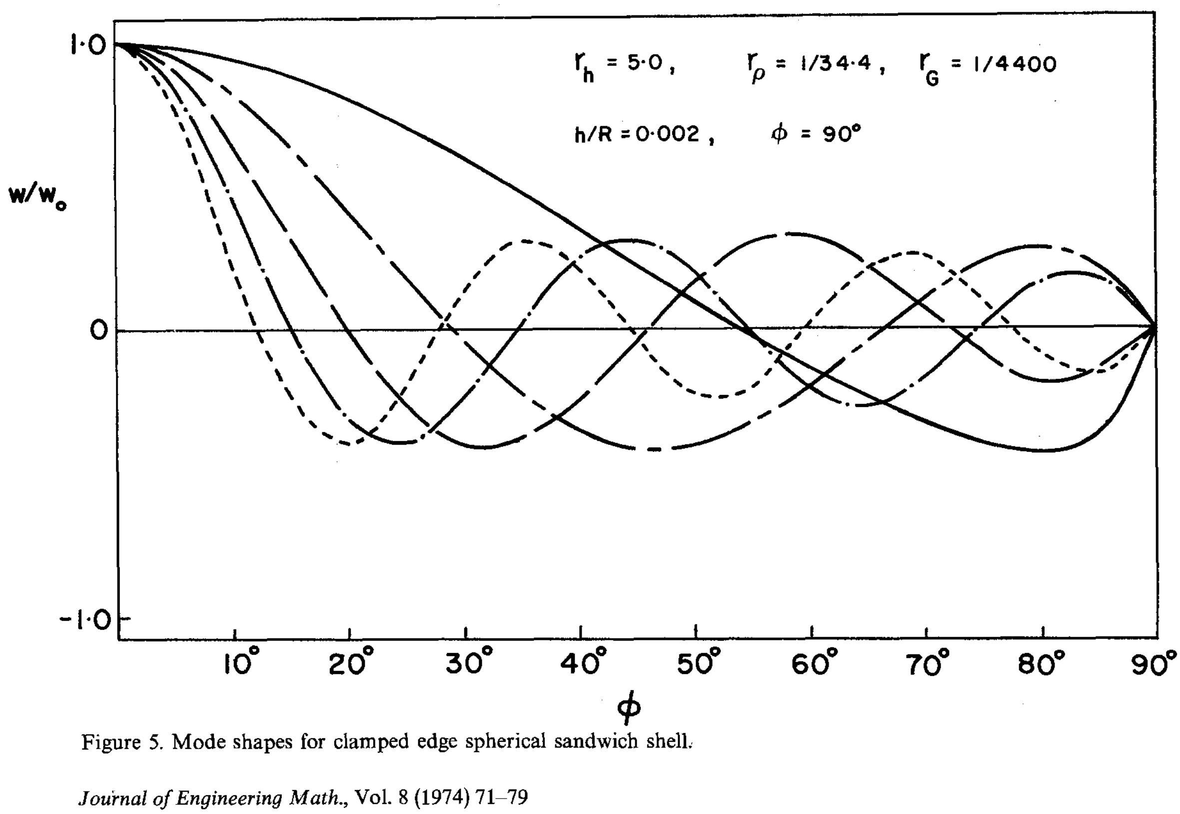

Figure 5. Mode shapes for clamped edge spherical sandwich shell.

where p_α is a real quantity thereby yielding real values of $P_{n_\alpha}(\cos\phi)$. When the two roots are complex conjugates, the resulting $P_{n_\alpha}(\cos\phi)$ are also complex conjugates. A plot of indices n_1, n_2, and n_3 as functions of Ω has been shown in Fig. 2.

As the values of Legendre functions are evaluated, the values of frequency determinant (4.2) are obtained for a series of values of Ω, keeping r_G, r_ρ, r_h, v, and h/R constant. The same procedure is repeated until value of the determinant changed the sign and then an accurate value of the root is generated.

The natural frequencies up to five modes for clamped edge hemispherical shell have been plotted with Ω as the ordinate and h/R as the abscissa as shown in Fig. 3. The lowest natural frequency increases very slowly and appears to be almost constant in Fig. 3. The frequency Ω increases with h/R for all modes, but at the same time the slope of the curve decreases. Detailed calculations for $\phi=60°$ to $\phi=90°$ were performed, but, due to space limitations, frequency variation is shown for $\phi=90°$.

In Figs. 4 and 5 the variations of w/w_0, where w_0 is the radial displacement at $\phi=0$, are shown with ϕ for opening angles 60° and 90° respectively. In these figures the plots of w/w_0 are examined up to five modes. The number of nodal circles in a particular mode is the same as the mode number. Within the framework of the present assumptions that the face sheets are membranes and the core is capable of taking shear deformations, it is improper to distinguish between the fixed and pinned conditions. In consequence, the slope $w_{,\phi}$ is irrelevant.

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